

DYNAMICS OF MODIFIED HOUSEHOLDER’S METHOD

Ashaq Ali

Department of Mathematics, Govt. College Kamoke, Gujranwala, Pakistan
arhamusman2001@gmail.com

Abdul Aziz Shahid and Muhammad Tanveer

Department of mathematics and statistics, The university of Lahore, Lahore Pakistan
abdulazizshahid@gmail.com, comtanveer.7955180@yahoo.com

Adnan Aslam

University of Engineering and Technology, Lahore Pakistan(RCET)
Adnanaslam15@yahoo.com

ABSTRACT: *The aim of this paper is to present a modification of the visualization process of finding the roots of a given complex polynomial. The images obtained are called polynomiographs. In this paper a modified Householder’s method is used to generate the polynomiographs of different complex polynomials. Polynomiographs have importance for both the art and science aspects. Presented examples show that we obtain very interesting patterns for complex polynomial equations. We believe that the results of this paper enrich the functionality of the existing polynomiography software.*

Key words: *Polynomials, Iterative method, Fractals, Polynomiographs.*

INTRODUCTION

Polynomials are one of the most significant objects in many fields of mathematics. Polynomial root-finding has played a key role in the history of mathematics. It is one of the oldest and most deeply studied mathematical problems. The last interesting contribution to the polynomials root finding history was made by Kalantari [16,17], who introduced the polynomiography. As a method which generates nice looking graphics, it was patented by Kalantari in USA in 2005 [17,18]. Polynomiography is defined to be “the art and science of visualization in approximation of the zeros of complex polynomials, via fractal and non fractal images created using the mathematical convergence properties of iteration functions” [16]. An individual image is called a “polynomiograph”. Polynomiography combines both art and science aspects. Polynomiography gives a new way to solve the ancient problem by using new algorithms and computer technology. Polynomiography is based on the use of one or an infinite number of iteration methods formulated for the purpose of approximation of the root of polynomials e.g. Newton’s method , Halley’s method etc. The word “fractal”, which partially appeared in the definition of polynomiography, was coined by the famous mathematician Benoit Mandelbrot [15]. Both fractal images and polynomiographs can be obtained via different iterative schemes. Fractals are self-similar has typical structure and independent of scale. On the other hand, polynomiographs are quite different. The “polynomiographer” can be controlled the shape and designed in a more predictable way by using different iteration methods to the infinite variety of complex polynomials. Generally, fractals and polynomiographs belong to different classes of graphical objects.

Polynomiography has diverse applications in math, science, education, art and design. According to Fundamental Theorem of Algebra, any complex polynomial with complex coefficients $\{a_n, a_{n-1}, \dots, a_1, a_0\}$

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 \tag{1}$$

of degree n has n roots (zeros) which may or may not be distinct. The degree of polynomial describes the number of

basins of attraction and placing roots on the complex plane manually localization of basins can be controlled.

Usually, polynomiographs are colored based on the number of iterations needed to obtain the approximation of some polynomial root with a given accuracy and a chosen iteration method. The description of polynomiography, its theoretical background and artistic applications are described in [16,17,18].

ITERATION

During the last century, the different numerical techniques for solving nonlinear equation $f(x) = 0$ have been successfully applied. For examples see [1-8, 12-14] and the reference therein. Now we define:

For a given x_0 , compute the approximate solution x_{n+1} by the following iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f^2(y_n)f''(y_n)}{2f'^3(y_n)}.$$

This is so-called modified Householder method for solving nonlinear equations. Let $p(z)$ be the complex polynomial, then

$$y_n = z_n - \frac{p(z_n)}{p'(z_n)} - \frac{p^2(z_n)p''(z_n)}{2p'^3(z_n)},$$

$$z_{n+1} = y_n - \frac{p(y_n)}{p'(y_n)} - \frac{p^2(y_n)p''(y_n)}{2p'^3(y_n)}.$$

where $z_0 \in \mathbb{C}$ is a starting point, is so-called modified two step AM method for solving nonlinear complex equations.

The sequence $\{z_n\}_{n=0}^\infty$ is called the orbit of the point z_0 converges to a root z^* of p then, we say that z_0 is attracted to z^* . A set of all such starting points for which $\{z_n\}_{n=0}^\infty$ converges to root z^* is called the basin of attraction of z^* .

APPLICATIONS

The applications of the predictor-corrector iterative method for solving nonlinear complex equations perturbs the shape of polynomial basins and makes the polynomiographs look more "fractal". The aim of using the predictor-corrector iterative method for solving nonlinear complex equations to create images that are quite new, different from images by the Newton's method and Householder's method free from second derivatives [10] and interesting from the aesthetic point of view.

In this section we present some examples of polynomiographs for different complex polynomials equation $p(z) = 0$. The different colors of images depend upon number of iterations to reach a root with given accuracy $\epsilon = 0.001$. One can obtain infinitely many nice looking polynomiographs by changing parameter k , where k is the upper bound of the number of iterations.

Polynomiograph for $z^3 - 1 = 0$

Complex polynomial equation $z^3 - 1 = 0$, having three roots: $1, -\frac{1}{2} - \frac{\sqrt{3}}{2}I, -\frac{1}{2} + \frac{\sqrt{3}}{2}I$. The polynomiograph is presented in the following figure with three distinct basins of attraction to the three roots of the polynomial $z^3 - 1 = 0$.

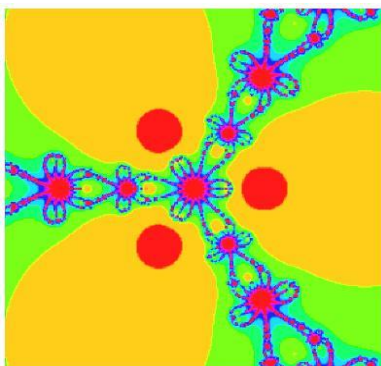


Fig. 1. Polynomiography for $z^3 - 1 = 0$.

Polynomiograph for $z^3 - 3z + 3 = 0$

Complex polynomial equation $z^3 - 3z + 3 = 0$, having three roots. The polynomiograph is presented in the following figure with three distinct basins of attraction to the three roots of the polynomial $z^3 - 3z + 3 = 0$.

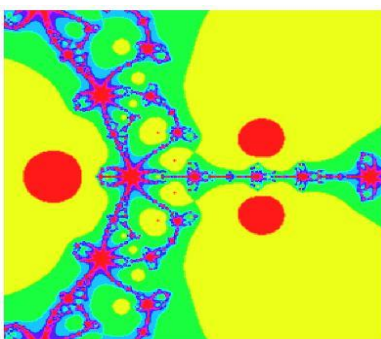


Fig. 2. Polynomiography for $z^3 - 3z + 3 = 0$.

Polynomiograph for $z^4 - 1 = 0$

Complex polynomial equation $z^4 - 1 = 0$, having four roots: $1, -1, I, -I$. The polynomiograph is presented in the following figure with four distinct basins of attraction to the four roots of the polynomial $z^4 - 1 = 0$.

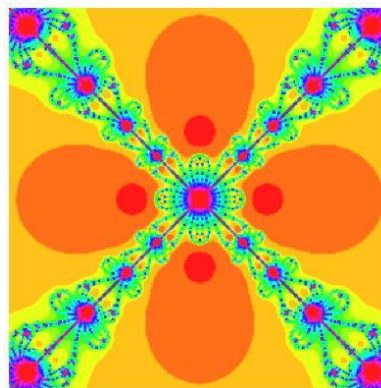


Fig. 3. Polynomiography for $z^4 - 1 = 0$.

Polynomiograph for $z^4 + 4 = 0$

Complex polynomial equation $z^4 + 4 = 0$, having four roots. The polynomiograph is presented in the following figure with four distinct basins of attraction to the four roots of the polynomial $z^4 + 4 = 0$.

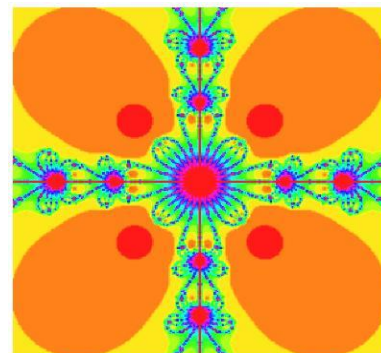


Fig. 4. Polynomiography for $z^4 + 4 = 0$.

Polynomiograph for $z^4 + z^3 - 1$

Complex polynomial equation $z^4 + z^3 - 1$ having four roots. The polynomiograph is presented in the following figure with four distinct basins of attraction to the four roots of the polynomial.

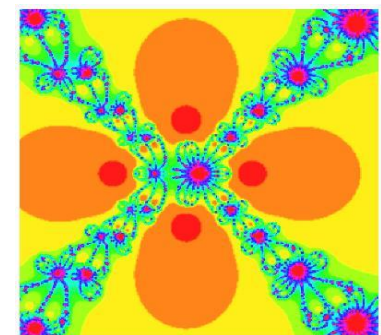


Fig. 5. Polynomiography for $z^4 + z^3 - 1$

Polynomiograph for $z^5 + z = 0$

Complex polynomial equation $z^5 + z$, having five roots. The polynomiograph is presented in the following figure with five distinct basins of attraction to the five roots of the polynomial $z^5 + z = 0$.

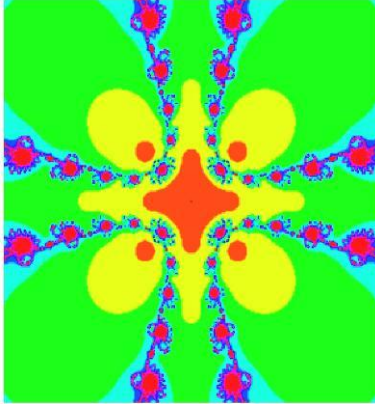


Fig. 6.Polynomiograph for $z^5 + z = 0$.

Polynomiograph for $z^6 - 1$

Complex polynomial equation $z^6 - 1$ having six roots. The polynomiograph is presented in the following figure with six distinct basins of attraction to the six roots of the polynomial.

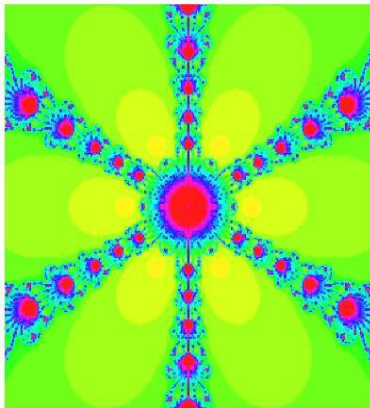


Fig. 7. Polynomiograph for $z^6 - 1$.

CONCLUSIONS

We presented some examples of polynomiographs for different complex polynomials equation $p(z) = 0$. We used the modified Householder’s method for solving nonlinear complex equations to create images that are quite new, different from images by the Newton’s method and Householder’s method free from second derivatives [10], and interesting from the aesthetic point of view.

REFERENCES

[1] A. Ali, M. S. Ahmad, W. Nazeer and M. Tanveer, *New modified two-step Jungck iterative method for solving nonlinear functional equations*. Sci.Int. (Lahore), 27(4), 2959-2963, 2015. [2] A. Ali, Q. Mehmood, M. Tanveer, A. Aslam and W. Nazeer, *Modified new third-order*

iterative method for nonlinear equations. Sci.Int.(Lahore), 27(3), 1741-1744, 2015.

[3] A. Ali, M. S. Ahmad, M. Tanveer, Q. Mehmood and W. Nazeer, *Modified two-step fixed point iterative method for solving nonlinear functional equations*, Sci.Int. (Lahore), 27(3), 1737-1739, 2015.

[4] M. S. Ahmad, A. Ali, M. Tanveer, A. Aslam and W. Nazeer, *New fixed point iterative method for solving nonlinear functional equations*. Sci.Int.(Lahore), 27(3), 1815-1817, 2015.

[5] A. Ali, W. Nazeer, M. Tanveer, M. Ahmad., *Modified Golbabi and Javidi’s method (MGJM) for solving nonlinear functions with convergence of order six*. Sci.Int. (Lahore), 28(1), 89-93, 2015

[6] W. Nazeer, S. M. Kang, M. Tanveer and A. A. Shahid, *Modified Two-step Fixed Point Iterative Method for Solving Nonlinear Functional Equations with Convergence of Order Five and Efficiency Index 2.2361*. Wulfina Journal. Vol 22, No. 5; May 2015.

[7] W. Nazeer, S. M. Kang and M. Tanveer, *Modified Abbasbandy’s Method for Solving Nonlinear Functions with Convergence of Order Six*, International Journal of Mathematical Analysis Vol. 9, 2015, no.41, 2011-2019

[8] W. Nazeer, A. Naseem, S. M. Kang and Y. C. Kwun *Generalized Newton Raphson’s method free from second derivative*, J. Nonlinear Sci. Appl. (to appear)

[9] W. Nazeer, M. Tanveer, S. M. Kang and Y. C. Kwun, *New third-order Fixed point iterative method for solving nonlinear functional equations and Polynomiography*, J. Nonlinear Sci. Appl. (to appear)

[10] W. Nazeer., M. Tanveer., S. M. Kang., and A. Naseem., *A new Householder’s method free from second derivatives for solving nonlinear equations and polynomiography*. J. Nonlinear Sci. Appl. 9 (2016), 998-1007

[11] S. M. Kang., S. M. Ramay., M. Tanveer., and W. Nazeer., *Polynomiography via an iterative method corresponding to Simpson’s $\frac{1}{3}$ rule*. J. Nonlinear Sci. Appl. 9 (2016), 967-976

[12] S. M. Kang, W. Nazeer, A. Rafiq and C. Y. Youg, *A new third order iterative method for scalar nonlinear equations*. Int. Journal of Math. Analysis, Vol. 8, 2014, no. 43, 2141 – 2150

[13] S. M. Kang, W. Nazeer, M. Tanveer, Q. Mehmood and K. Rehman, *Improvements in Newton-Rapshon Method for Nonlinear Equations Using Modified Adomian Decomposition Method*. International Journal of Mathematical Analysis, Vol.9, 2015, no.39, 1919-1928.

[14] M. S. Khan, A. Nazir, W. Nazeer., *Iterative method for solving nonlinear functions with convergence of order four*. Sci.Int. (Lahore), 28(1), 77-81, 2016.

[15] B. Kalantari, *Polynomial Root-Finding and Polynomiography*, World Scientific, Singapore, 2009.

[16] B. Kalantari, *Method of creating graphical works based on polynomials*, U.S. Patent 6,894--705, 2005.

[17] B. Kalantari, *Polynomiography: from the fundamental theorem of Algebra to art*, Leonardo, vol. 38, no. 3, pp. 233-238, 2005.

- [18] W. Kotarski, K. Gdawiec, and A. Lisowska, *Polynomiography via Ishikawa and Mann iterations*, Advances in Visual Computing, Part I, G. Bebis, R. Boyle, B. Parvin et al., Eds., vol. 7431 of Lecture Notes in Computer Science, pp. 305--313, Springer, Berlin, Germany, 2012.
- [19] M. Tanveer, M. Ahmad, A. Ali, W. Nazeer, K. Rehman., *Modified Householder's method (MHHM) for solving nonlinear functions with convergence of order six*. Sci.Int.(Lahore), 28(1), 83-87, 2016.